

# Numerical Modeling of Coexistence, Competition and Collapse of Rotating Spiral Waves in Three-Level Excitable Media with Discrete Active Centers and Absorbing Boundaries

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Spatio-temporal dynamics of excitable media with discrete three-level active centers (ACs) and absorbing boundaries is studied numerically by means of a deterministic three-level model (see S. D. Makovetskiy and D. N. Makovetskii, <http://arxiv.org/abs/cond-mat/0410460> ), which is a generalization of the Zykov-Mikhailov model (see Sov. Phys. – Doklady, 1986, Vol.31, No.1, P.51) for the case of two-channel diffusion of excitations. In particular, we revealed some qualitatively new features of coexistence, competition and collapse of rotating spiral waves (RSWs) in three-level excitable media under conditions of strong influence of the second channel of diffusion. Part of these features are caused by unusual mechanism of RSWs evolution when RSW's cores get into the surface layer of an active medium (i. e. the layer of ACs resided at the absorbing boundary). Instead of well known scenario of RSW collapse, which takes place after collision of RSW's core with absorbing boundary, we observed complicated transformations of the core leading to nonlinear “reflection” of the RSW from the boundary or even to birth of several new RSWs in the surface layer. To our knowledge, such nonlinear “reflections” of RSWs and resulting die hard vorticity in excitable media with absorbing boundaries were unknown earlier.

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## I. INTRODUCTION

Rotating spiral waves (RSWs) are typical robust self-organized structures (autowaves) in various dissipative systems, ranging from chemistry, biology and ecology [7, 16, 32] to nonlinear optics and lasers [49].

Evolution of these spatio-temporal structures is now a subject of extensive theoretical studies, experimental investigations and computer modeling (see, e.g., [19, 37, 40, 44, 47] and references therein).

Many mathematical models in this field are based on concept of excitability [35]. An important approach in modern computer investigations of excitable systems is based on using of discrete parallel models with local interactions, mostly by cellular automata (CA).

In the framework of the CA approach [3, 53], the three-level representation of the excitable media states is an effective tool for carrying computer experiments with multi-particle systems possessing long-time nonstationary dynamics (see [28] and references therein). Parallelization of updating of the active medium states and locality of interactions between elementary parts of the medium are the fundamental principles of the CA approach. Using of this approach admits to fulfill modeling of large systems by emulation of parallel matrix transformations.

The essence of three-level representation of excitable medium [28, 51, 56] is as follows. Each elementary part or active center (AC) of an excitable system has the single stable *ground* state, say  $L_I$ , and at least two metastable (upper) states, say  $L_{II}$  and  $L_{III}$ .

The higher metastable state  $L_{III}$  is the *excited* one, it may be reached only after certain hard perturbation (from neighboring ACs or from external source) of the ground state  $L_I$  of the considered AC. A group of excited ACs may excite another ACs etc. Due to metastability, the lifetime  $\tau_e$  of the excited state  $L_{III}$  is a finite value, after which every excited AC spontaneously reaches the intermediate *refractory* state  $L_{II}$ .

Refractoriness means “sleeping” state at which no excitation of the considered AC is possible both from neighboring ACs or from an external source. Moreover, the AC in refractory state can't participate in excitation of another ACs (lying at  $L_I$ ). So, after a refractoriness lifetime  $\tau_r$ , the considered AC reaches the ground state  $L_I$ . Usually only the cyclic transitions ( $L_I \rightarrow L_{III} \rightarrow L_{II} \rightarrow L_I \rightarrow \dots$ ) are permitted.

The concept of excitability was developed in biology, but it is widely used now in chemistry and physics. In particular, this concept have been applied to solve many problems arising in modern nonlinear optics, laser physics etc. [10, 11, 12, 39, 43, 48, 50]

In [28] we have considered even more close laser-like analog of excitable system, namely the three-level active medium of the microwave phonon laser (phaser) with dipole-dipole interactions between AC. The only signifi-

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cant difference of phaser medium from the usual excitable one is the second channel of diffusion of excitations. This system was experimentally studied in Institute of Radio-Physics and Electronics (Kharkov, Ukraine) [22, 23, 24, 25, 26, 27], demonstrating self-organization, bottlenecked cooperative transient processes and other nonlinear phenomena under extremely low level of intrinsic quantum noise (phaser has 15 orders lower intensity of spontaneous emission in comparison to usual optical-range lasers).

In preceding publications [28, 29, 30, 31] we carried out a series of computer experiments on RSWs dynamics in autonomous excitable systems with various control parameters and random initial excitations. The most interesting phenomena of self-organized vorticity observed in computer experiments [28, 29, 30, 31] were as follows: (a) spatio-temporal transient chaos in form of highly bottlenecked collective evolution of excitations by RSWs with variable topological charges; (b) competition of left-handed and right-handed RSWs with unexpected features, including self-induced alteration of integral effective topological charge; (c) transient chimera states, i. e. coexistence of regular and chaotic domains in excitable media; (d) branching of an excitable medium states with different symmetry which may lead to full restoring of symmetry of imperfect starting pattern. Phenomena (a) and (c) are directly related to microwave phonon laser dynamics features observed earlier in real experiments at liquid helium temperatures on corundum crystals doped by iron-group ions [22, 23, 24, 25, 26, 27].

In the present paper, we report some qualitatively new features of coexistence, competition and collapse of RSWs in three-level model [28, 29, 30, 31] of excitable media under conditions of strong influence of the second channel of diffusion of excitations. Part of these features are caused by unusual mechanism of RSWs evolution when RSW's cores get into the surface layer of an active medium (i. e. the layer of ACs resided at the absorbing boundary). Instead of well known scenario of RSW collapse, which takes place after collision of RSW's core with absorbing boundary, we observed complicated transformations of the core leading to "reflection" of the RSW from the boundary or even to birth of several new RSWs in the surface layer. To our knowledge, such nonlinear "reflections" of RSWs and resulting die hard vorticity in excitable media with absorbing boundaries were unknown earlier.

## II. SOME REMARKS ON ZYKOV-MIKHAIEV MODEL AND ITS TWO-CHANNEL MODIFICATION

The Zykov-Mikhailov (ZM) model [56] is a discrete parallel mapping with local interactions, which may be defined as three-level CA [21]. Let an active discretized medium  $\mathfrak{M}_e$  has the form of rectangular 2D-lattice. Each cell of the lattice contains the single three-level AC with

coordinates  $(i, j)$ , where  $(\min(i) = 1), (\min(j) = 1)$ . All the ACs in the  $\mathfrak{M}_e$  are identical, and they interact by the same set of rules (ZM model, as well as most of the CA models, is homogeneous and isotropic in the von Neumann sense [36]). Each AC has the single stable *ground* state  $L_I$ , and two metastable states  $L_{II}$  and  $L_{III}$ .

The upgrade of states  $S_{ij}^{(n)} \equiv S^{(n)}(i, j)$  of all ACs (i. e. their intralevel or interlevel transitions) are carrying out synchronously at each step  $0 < n \leq N$  during the system evolution. All intralevel ( $L_I \rightarrow L_I, L_{II} \rightarrow L_{II}, L_{III} \rightarrow L_{III}$ ) and some of interlevel (namely  $L_I \rightarrow L_{III}, L_{III} \rightarrow L_{II}, L_{II} \rightarrow L_I$ ) transitions are permitted. The final step  $N$  may be either predefined or it will be searched during CA evolution as time (i.e. quantity of discrete steps) for reaching an attractor of this CA. The conditions of upgrade [21, 56] depend both on  $S_{ij}^{(n-1)}$  and on  $S_{i'j'}^{(n-1)}$ , where  $\{i', j'\}$  belongs to certain active neighborhood of AC at site  $(i, j)$ . In the framework of the ZM model (and in most of other CA models of excitable chemical systems), diffusion of excitations is possible only by  $L_I \rightarrow L_{III}$  transition. This is a single-channel (1C) mechanism of diffusion.

A modification of ZM model was proposed by us [28, 29, 30, 31] to adapt it for emulation of some aspects of dynamics of class-B optical lasers [43, 48, 50] and microwave phonon lasers [22, 23, 24, 25, 26, 27]. The relaxational properties of ACs in microwave phonon lasers are of the same type, as in class-B optical lasers [45], and there are experimental confirmations [23, 24, 25, 26] of common properties of microwave phonon lasers and class-B optical lasers.

In order to formulate boundary conditions for a  $\mathfrak{M}_e$  of finite size (i. e. when  $\max(i) = M_X, \max(j) = M_Y$ ), a set  $\mathfrak{M}_v$  of virtual cells with coordinates  $(i = 0) \vee (j = 0) \vee (i = M_X + 1) \vee (j = M_Y + 1)$  may be introduced [28, 29, 30, 31]. Each virtual cell contains the unexcitable center having the single level  $L_0$ . The surface layer of a bounded active system is represented by the set  $\mathfrak{M}_{\text{surf}}$  of ACs with coordinates  $(i = 1) \vee (j = 1) \vee (i = M_X) \vee (j = M_Y)$ . Note that our surface layer is of 1D type because active medium itself is of 2D type (we stress this circumstance to avoid misunderstandings).

The original ZM model has, as it was pointed out earlier, the single channel of diffusion [21, 56]. Such 1C-diffusion models are adequate for chemical reaction-diffusion systems [46]. In microwave phonon laser active system, the multichannel diffusion of spin excitations is the typical case, because it proceeds via (near)-resonant dipole-dipole ( $d - d$ ) magnetic interactions between paramagnetic ions [22]. For a three-level system, which is the simplest microwave phonon laser system, there are 3 possible channels of resonant diffusion. In the case when  $d - d$  interactions are forbidden at one of three resonant frequencies of a three-level system, the asymmetric two-channel (2C) diffusion is realized under conditions of perfect refractoriness of the intermediate level — see [28, 29, 30, 31]. Note that asymmetric diffusion is well known in biology (see, e.g., the book of D. A. Frank-

Kamenetzky [9]). Recently a kind of asymmetric diffusion was proposed by N. Packard and R. Shaw [38] for a mechanical system.

### III. THREE-LEVEL MODEL OF EXCITABLE SYSTEM WITH TWO-CHANNEL DIFFUSION (A MODIFIED ZYKOV-MIKHAIEV MODEL)

#### A. States of active centers and branches of evolution operator

Let  $\mathfrak{M}_e$  is a rectangular 2D lattice containing  $M_X \times M_Y$  three-level ACs. The upgrade of state  $S_{ij}^{(n)}$  of each AC is carried out synchronously during the system evolution. The excited (upper) level  $L_{\text{III}}$  has the time of relaxation  $\tau_e \geq 1$ , and the refractory (intermediate) level  $L_{\text{II}}$  has the time of relaxation  $\tau_r \geq 1$ . Both  $\tau_e$  and  $\tau_r$  are integer numbers. This model of excitable system is a kind of three-level CA. The original ZM model [21, 56] was formulated as three-level CA with 1C diffusion.

In our three-level model of excitable system (TLM), the second channel of diffusion of excitations is included. A detailed argumentation for using of the 2C diffusion mechanism one can find in [28].

In the TLM, the first channel of diffusion accelerates the transitions  $L_I \rightarrow L_{\text{III}}$  for a given AC, and the second channel of diffusion (which is absent in the ZM model) accelerates the transitions  $L_{\text{III}} \rightarrow L_{\text{II}}$ . The complete description of AC's states  $S_{ij}^{(n)}$  in such the TLM model includes one type of global attributes (the phase counters

$\varphi_{ij}^{(n)}$ ) and two types of partial attributes  $u_{ij}^{(n)}$  and  $z_{ij}^{(n)}$  for each individual AC in  $\mathfrak{M}_e$ . Full description of all AC's possible states is as follows:

$$S_{ij}^{(n)}(L_I) = (\varphi_{ij}^{(n)}, u_{ij}^{(n)}); \quad (1)$$

$$S_{ij}^{(n)}(L_{\text{III}}) = \begin{cases} (\varphi_{ij}^{(n)}), & \text{if } n = 0, \\ (\varphi_{ij}^{(n)}, z_{ij}^{(n)}), & \text{if } n \neq 0; \end{cases} \quad (2)$$

$$S_{ij}^{(n)}(L_{\text{II}}) = (\varphi_{ij}^{(n)}). \quad (3)$$

In the framework of the TLM model, the phase counters lie in the interval  $\varphi_{ij}^{(n)} \in [0, \tau_e + \tau_r]$ . The following correspondences between  $\varphi_{ij}^{(n)}$  and  $L_K$  take place (by definition) for all the ACs in  $\mathfrak{M}_e$  at all steps  $n$  of evolution:

$$(L_K = L_I) \Leftrightarrow (\varphi_{ij}^{(n)} = 0); \quad (4)$$

$$(L_K = L_{\text{III}}) \Leftrightarrow (0 < \varphi_{ij}^{(n)} \leq \tau_e); \quad (5)$$

$$(L_K = L_{\text{II}}) \Leftrightarrow (\tau_e < \varphi_{ij}^{(n)} \leq \tau_e + \tau_r). \quad (6)$$

These correspondences are of key significance for all multistep-relaxation models (Reshodko model [4, 5, 41], ZM model [21, 56] etc.): there are only three discrete levels, and relaxation of each AC is considered as intralevel transitions (or transitions between virtual sublevels).

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The evolution of each individual AC proceeds by sequential cyclic transitions  $L_K \rightarrow L_{K'}$  (where  $K$  and  $K' \in \{I, \text{III}, \text{II}\}$ ), induced by the Kolmogorov evolution operator  $\widehat{\Omega}$  [17]. In the TLM model, the evolution operator  $\widehat{\Omega}$  has three orthogonal branches  $\widehat{\Omega}_I$ ,  $\widehat{\Omega}_{\text{III}}$  and  $\widehat{\Omega}_{\text{II}}$ , which we call ground, excited and refractory branches respectively. The choice of the branch at iteration  $n+1$  is dictated only by the global attribute of the AC at step  $n$ , namely:

$$\varphi_{ij}^{(n+1)} = \widehat{\Omega}(\varphi_{ij}^{(n)}) = \begin{cases} \widehat{\Omega}_I(\varphi_{ij}^{(n)}), & \text{if } \varphi_{ij}^{(n)} = 0, \\ \widehat{\Omega}_{\text{III}}(\varphi_{ij}^{(n)}), & \text{if } 0 < \varphi_{ij}^{(n)} \leq \tau_e, \\ \widehat{\Omega}_{\text{II}}(\varphi_{ij}^{(n)}), & \text{if } \tau_e < \varphi_{ij}^{(n)} \leq \tau_e + \tau_r. \end{cases} \quad (7)$$

#### B. Ground branch of the evolution operator

At step  $n+1$ , the branch  $\widehat{\Omega}_I$  by definition fulfills operations only over those ACs, which have  $L_K = L_I$  at step  $n$ . These operations are precisely the same, as in ZM model [21, 56], namely:

$$\varphi_{ij}^{(n+1)} = \widehat{\Omega}_I(\varphi_{ij}^{(n)}) = \begin{cases} 0, & \text{if } (\varphi_{ij}^{(n)} = 0) \wedge (u_{ij}^{(n+1)} < h); \\ 1, & \text{if } (\varphi_{ij}^{(n)} = 0) \wedge (u_{ij}^{(n+1)} \geq h), \end{cases} \quad (8)$$

$$u_{ij}^{(n+1)} = A_{ij}^{(n)}(S_{ij}^{(n)}) + D_{ij}^{(n)}(S_{i+p,j+q}^{(n)}) = g u_{ij}^{(n)} + \sum_{p,q} C(p, q) J_{i+p,j+q}^{(n)}, \quad (9)$$

where  $A_{ij}^{(n)}$  is the accumulating term for the  $u$ -agent, which is an analog of chemical activator [21, 56];  $D_{ij}^{(n)}$  is the first-channel diffusion term;  $h$  is the threshold for the  $u$ -agent ( $h > 0$ );  $g$  is the accumulation factor for the  $u$ -agent ( $g \in [0, 1]$ );  $C(p, q)$  is the active neighborhood of the AC at site  $(i, j)$ ; and the  $u$ -agent arrives to ACs with  $L_K = L_I$  only from ACs with  $L_K = L_{\text{III}}$  in  $C(p, q)$ :

$$J_{i+p,j+q}^{(n)} = \begin{cases} 1, & \text{if } (0 < \varphi_{i+p,j+q}^{(n)} \leq \tau_e); \\ 0, & \text{if } (\varphi_{i+p,j+q}^{(n)} > \tau_e) \vee (\varphi_{i+p,j+q}^{(n)} = 0). \end{cases} \quad (10)$$

The definition of the diffusion term  $D_{ij}^{(n)}$  in Eqn. (9) is very flexible. Apart of well-known neighborhoods of the Moore ( $C(p, q) = C_M(p, q)$ ) and the von Neumann ( $C(p, q) = C_N(p, q)$ ) types:

$$C_M(p, q) = \begin{cases} 1, & \text{if } [(|p| \leq 1) \wedge (|q| \leq 1)] \wedge (\delta_{p0}\delta_{q0} \neq 1); \\ 0, & \text{otherwise,} \end{cases} \quad (11)$$

$$C_N(p, q) = \begin{cases} 1, & \text{if } [(|r| = 1) \oplus (|s| = 1)]; \\ 0, & \text{otherwise,} \end{cases} \quad (12)$$

one can easily define another neighborhoods, e. g. of the box type  $C_B(p, q)$  or of the diamond type  $C_D(p, q)$  [8] (which are the straightforward generalization of the Moore  $C_M(p, q)$  and the von Neumann  $C_N(p, q)$  neighborhoods), etc.

On the other hand, the ZM definition [21, 56] of the weight factors  $J_{i+p,j+q}^{(n)}$  (10) may be extended out of the binary set  $\{0, 1\}$  to model various distance-dependent phenomena within, e. g.,  $C_D(p, q), C_B(p, q)$ .

In this work, however, we restricted ourselves by the Moore neighborhood and by weight factors of the form (10). Another types of neighborhoods, weight factors and some other modifications of the model will be studied in subsequent papers.

### C. Excited branch of the evolution operator

At the same step  $n + 1$ , the branch  $\widehat{\Omega}_{\text{III}}$  fulfills operations only over those ACs, which have  $L_K = L_{\text{III}}$  at step  $n$ :

$$\varphi_{ij}^{(n+1)} = \widehat{\Omega}_{\text{III}}(\varphi_{ij}^{(n)}) = \begin{cases} \varphi_{ij}^{(n)} + 1, & \text{if } [(0 < \varphi_{ij}^{(n)} < \tau_e) \wedge (z_{i,j}^{(n+1)} < f)] \vee (\varphi_{ij}^{(n)} = \tau_e); \\ \varphi_{ij}^{(n)} + 2, & \text{if } (0 < \varphi_{ij}^{(n)} < \tau_e) \wedge (z_{i,j}^{(n+1)} \geq f). \end{cases} \quad (13)$$

$$z_{ij}^{(n+1)} = \overline{D}_{ij}^{(n)}(S_{i+p,j+q}^{(n)}) = \sum_{p,q} C(p, q) Q_{i+p,j+q}^{(n)}. \quad (14)$$

where  $\overline{D}_{ij}^{(n)}$  is the second-channel diffusion term;  $f$  is the threshold for the  $z$ -agent ( $f > 0$ ), and we assume that  $z$ -agent arrives to excited AC ( $L_K = L_{\text{III}}$ ) only from those ACs, which have  $L_K = L_I$  in  $C(p, q)$ :

$$Q_{i+p,j+q}^{(n)} = \begin{cases} 1, & \text{if } \varphi_{i+p,j+q}^{(n)} = 0; \\ 0, & \text{if } \varphi_{i+p,j+q}^{(n)} \neq 0. \end{cases} \quad (15)$$

One can see from (14) that the  $z$ -agent does not accumulate during successive iterations. In other words, the branch  $\widehat{\Omega}_{\text{III}}$  at step  $n + 1$  produces "memoryless" values of partial attributes  $z_{ij}^{(n+1)}$  for ACs having  $L_K = L_{\text{III}}$  at step  $n$  (in contrary to the  $u$ -agent for ACs having  $L_K = L_I$  at step  $n$ ). The  $z$ -agent may accelerate transitions from excited ACs to refractory ones. This is the most important difference between our 2C model of three-level excitable medium and the original 1C model of ZM [21, 56].

#### D. Refractory branch of the evolution operator

The branch  $\widehat{\Omega}_{\text{II}}$  does not produce/change any partial attributes at all (because the intermediate level  $L_{\text{II}}$  is in the state of refractoriness). It fulfils the operations only over those ACs, which have  $L_K = L_{\text{II}}$  at step  $n$ :

$$\varphi_{ij}^{(n+1)} = \widehat{\Omega}_{\text{II}}(\varphi_{ij}^{(n)}) = \begin{cases} \varphi_{ij}^{(n)} + 1, & \text{if } \tau_e < \varphi_{ij}^{(n)} < \tau_e + \tau_r; \\ 0, & \text{if } \varphi_{ij}^{(n)} = \tau_e + \tau_r. \end{cases} \quad (16)$$

Generally speaking, there are many examples of active media with weak refractoriness (when the unit at the intermediate level  $L_{\text{II}}$  is not absolutely isolated from its neighborhood units). But in this work we restrict ourselves by the case of perfect refractoriness (16), which is valid, e. g., for the microwave phonon laser systems of the  $\text{Ni}^{2+} : \text{Al}_2\text{O}_3$  type [28].

#### E. Reduction of the TLM model to the ZM automaton

The second-channel diffusion gives the contribution to the TLM dynamics if  $f \leq \mathcal{N}$ , i.e. if  $f \leq 4$  for  $C = C_N$ ,  $f \leq 8$  for  $C = C_M$  and so on.

At  $((C = C_N) \wedge (f > 4))$  our TLM model is of ZM-like (i.e. 1C) type, and at  $((C = C_M) \wedge (f > 8))$  it becomes equivalent to the original ZM model [21, 56], which (with slight rearrangement of cases) is as follows:

$$\varphi_{ij}^{(n+1)} = \begin{cases} \varphi_{ij}^{(n)} + 1, & \text{if } 0 < \varphi_{ij}^{(n)} < \tau_e + \tau_r; \\ 1, & \text{if } (\varphi_{ij}^{(n)} = 0) \wedge (u_{ij}^{(n+1)} \geq h); \\ 0, & \text{if } [(\varphi_{ij}^{(n)} = 0) \wedge (u_{ij}^{(n+1)} < h)] \vee (\varphi_{ij}^{(n)} = \tau_e + \tau_r), \end{cases} \quad (17)$$

where  $u_{ij}^{(n)}$  is defined as in (9)–(11).

#### F. Geometry of active media, boundary conditions and transients in TLM

Pattern evolution is very sensitive to geometry of active medium and boundary conditions even in the framework of operation of the same CA. There are three types of geometry and boundary condition (GBC) combinations most commonly used in computer experiments with CA (see Table I). Let us consider them in details.

- The GBC-1 type is defined as follows:

$$S^{(n)}(i_v, j_v) = S^{(n)}(i_e^\Gamma, j_e^\Gamma), \quad (18)$$

where

$$i_e^\Gamma = \begin{cases} M_X & \text{if } i_v = 0; \\ 1, & \text{if } i_v = M_X + 1; \\ i_v, & \text{otherwise,} \end{cases} \quad (19)$$

$$j_e^\Gamma = \begin{cases} M_Y & \text{if } j_v = 0; \\ 1, & \text{if } j_v = M_Y + 1; \\ j_v, & \text{otherwise.} \end{cases} \quad (20)$$

Here  $(i_v, j_v) \in \mathfrak{M}_v$ , and the coordinates  $(i_e^\Gamma, j_e^\Gamma)$  belong to the border set  $\mathfrak{M}_e^\Gamma \subset \mathfrak{M}_e$  of excitable area, i. e.  $(i_e, j_e) = (i_e^\Gamma, j_e^\Gamma)$  if  $(i = 1) \vee (j = 1) \vee (i = M_X) \vee (j = M_Y)$ .

- The GBC-2 type can be easily defined by using an extended interval for the phase counter  $\varphi$ . Let every cell  $(i_v, j_v)$  in  $\mathfrak{M}_v$  contains one *unexcitable* unit. All these unexcitable units have frozen  $\varphi^{(n)}(i_v, j_v)$  at all  $n \in [0, N]$  without reference to states of  $(i_e^\Gamma, j_e^\Gamma)$ , and

$$\varphi^{(n)}(i_v, j_v) = \chi = \text{const}, \quad (21)$$

where,  $\chi > \tau_e + \tau_r$  or  $\chi < 0$ . These unexcitable units in  $\mathfrak{M}_v$  “absorb” excitations at the border of  $\mathfrak{M}_e$  in contrary to the case of GBC-1, where excitations at the border  $\mathfrak{M}_e$  are reinjected in active medium. In many cases this difference leads to qualitatively different behaviour of the whole TLM. Note that by this way one may also define TLM with *inhomogeneous* active medium, where some of ACs are changed to unexcitable units, i. e. unexcitable units are placed not only in  $\mathfrak{M}_v$ , but in  $\mathfrak{M}_e$  too. This may be easily done by introducing an additional orthogonal branch  $\widehat{\Omega}_0$  into the evolution operator  $\widehat{\Omega}$ . The pointed additional branch is activated at  $\varphi_{ij}^{(n)} = \chi$  only and is simply the identity operator  $\widehat{\Omega}_0 = \widehat{I}$ , i.e.:  $\varphi_{ij}^{(n+1)} = \widehat{\Omega}_0 \varphi_{ij}^{(n)} = \varphi_{ij}^{(n)}$ . And even more complex behaviour of such “impurity” units may be defined using analogous approach (i. e. by introducing additional orthogonal branches in an evolution operator): there may be pacemakers [21] or another special units embedded in active medium and described by phase counter  $\varphi \neq \chi$  in extended areas ( $\varphi > \tau_e + \tau_r$ ) or ( $\varphi < 0$ ).

TABLE I: Geometry of active media and boundary conditions

Type	Geometry	Boundary conditions
GBC-1	Bounded, Toroidal	Cyclic
GBC-2	Bounded, Flat	Zero-flow
GBC-3	Unbounded, Flat	(free space)

• The GBC-3 type is, strictly speaking, a case of borderless system. But the starting pattern has, of course, bounded quantity of ACs with  $L_K \neq L_I$ , located in bounded part of active medium. Hence, during the whole evolution ( $1 \leq n \leq N$ ) the front of growing excited area will meet the unexcited (but excitable!) ACs only. For the case GBC-3, there is neither absorption of excitations at boundaries (as for GBC-2), nor feedback by reinjection of excitations in the active system (as for GBC-1). From this point of view CA of GBC-3 type are “simpler”, than CA of GBC-1 and GBC-2 types. On the other hand, CA of GBC-3 type are potentially infinite discrete systems where true aperiodic (irregular, chaotic) motions are possible, in contrast to finite discrete systems, possessing, of course, only periodic trajectories in certain phase space after ending of transient stage [20].

On the other hand, there is also very important difference between GBC-2 and other two types of GBC. The system of GBC-2 type is the single from the three ones under consideration which interacts with the external world dynamically (besides of relaxation). Sure enough, GBC-1 and GBC-3 are connected to this outer world *only* by relaxation channels (the  $\tau_e$  and  $\tau_r$  are the measures of this connection). In contrary, a system of the GBC-2 type interacts with surroundings through the real boundary, which is in fact absent in toroidal finite-size active medium of GBC-1 type and it is absent by definition for a system of GBC-3 type. Despite of elementary mode of such interaction (boundary simply “absorbs” the outside-directed flow of excitations from  $(i_e^\Gamma, j_e^\Gamma)$ , a system of the GBC-2 type may demonstrate very special behaviour. The main attention in this work is devoted just to TLM of GBC-2 type as the most realistic model of microwave phonon laser active system, where both the mechanisms of interaction (dynamical and relaxational) of an active medium with the outer world are essential. Dissipation in a microwave phonon laser active medium (highly perfect single crystal at liquid helium temperatures) is caused by two main mechanisms: (a) dynamical, by coherent microwave phonon and photon emission directly through crystal boundaries and (b) relaxational, by thermal phonon emission. There are, of course, many more or less important differences between the TLM model and the real microwave phonon laser systems [23, 24, 25, 26], but in any case autonomous microwave phonon laser has only these two mechanisms of interaction with the outer world.

## G. Initial conditions

Patterns in ZM are described in terms of levels (or “colors”) of ACs, and initial conditions are formulated simply as matrix of ACs levels  $L_K$ . But states of ACs in such the automata as ZM or TLM are fully defined not only by levels  $L_K$  itself. There are global attributes which must be predefined before TLM evolution is started. Some of partial attributes must be predefined too. These points are essential for reproducing of the results of computer experiments with ZM, TLM, etc.

In our work such the initial conditions for the global attributes  $\varphi_{ij}$  are used:

$$\begin{cases} \varphi_{ij}^{(0)} = 0, & \text{if } L_K^{(0)} = L_I; \\ \varphi_{ij}^{(0)} = 1, & \text{if } L_K^{(0)} = L_{III}; \\ \varphi_{ij}^{(0)} = \tau_e + 1, & \text{if } L_K^{(0)} = L_{II}, \end{cases} \quad (22)$$

Initial conditions for  $u_{ij}$  must be defined for ground-state AC’s ( $L_K^{(0)} = L_I$ ) only. In this work, we suppose

$$(u_{ij}^{(0)} = 0) \quad \text{IFF} \quad (\varphi_{ij}^{(0)} = 0), \quad (23)$$

where IFF means “if and only if”. Initial conditions for  $z_{ij}$  are undefined for all  $L_K^{(0)}$  because  $z$ -agent is not defined at  $n = 0$ , see Eqn. (2). As the result, initial conditions will be defined in our TLM model as the matrix  $\|\varphi_{ij}^{(0)}\|$  (where  $\varphi_{ij}^{(0)} \in \{0, 1, \tau_e + 1\}$ ) with additional condition given by Eq. (23).

## IV. RESULTS AND DISCUSSION

A bounded solitary domain of excited ACs having appropriate relaxation times and placed far from grid boundaries (or at unbounded grid) may evolve to RSWs if and only if there is an adjoined (but not a surrounding) domain of refractory ACs [2, 8, 21, 42]. In this case RSWs appear by pair with opposite  $\text{sgn}(Q_T)$  and integral topological charge  $Q_T^{(\text{integr})} \equiv \sum_m Q_T^{(m)}$  is obviously conserved (by infinity in time if grid is unbounded). If such solitary excited-refractory area is placed in a starting pattern near the grid boundary and the GBC-2 conditions take place, the single RSW appears and evolves and  $Q_T^{(\text{integr})}$  is, of course, not conserved in this case. These simplest and well known examples illustrate possibilities of coexistence and competition of one or two RSWs in excitable media, and possible scenarios of the RSW(s) evolution may be easily forecasted.

An evolution of complex patterns with multiple, irregularly appearing, chaotically-like drifting and colliding RSWs (or, possibly, another spatio-temporal structures) is in essence unpredictable without direct computing of the whole transient stage. So, the best way to investigate cellular automaton (TLM in particular) is to run it, because, as S. Wolfram pointed out, “their own evolution

is effectively the most efficient procedure for determining their future" (see [52], page 737).

Here we present results of our computer experiments with 2C model of excitable media described in Section III. The main tool used in these experiments was the cross-platform software package "Three-Level Model of Excitable System" (TLM) © 2006 S. D. Makovetskiy [31]. It is based on extended and improved algorithms of discrete modeling of three-level many-particle excitable systems, proposed by us in [29, 30, 31]. The TLM package is written in the Java 2 language [15] using the Java 2 SDK (Standard Edition, version 1.4.2) and the Swing Library.

A discrete system with finite set of levels may have only two types of dynamically stable states at bounded lattice. The first of them is stationary state, and the second is periodic one. They may be called attractors by analogy with lumped dynamical systems (see e. g. [18, 54, 55]). For our cellular automaton, the first of such the attractors is spatially-uniform and time-independent state with  $\varphi_{ij}^{(n)} = 0$ , where  $(i \in [1, M_X]) \wedge (j \in [1, M_Y])$ ,  $n \geq n_C$ . In other words, the only stationary state of TLM is the state of full collapse of excitations at some step  $n_C$  (by definition of excitable system). The second type of TLM attractors includes many various periodically repeated states (RSW is a typical but not the single case). In this case  $\varphi_{ij}^{(n)} = \varphi_{ij}^{(n+T)}$  where  $(i \in [1, M_X]) \wedge (j \in [1, M_Y])$ ,  $n \geq n_P$ ;  $n_P$  is the first step of motion at a periodic attractor;  $T$  is the integer-number period of this motion,  $T > 1$ .

If starting patterns are generated with random spatial distribution by levels, then the time intervals  $n_C$  or  $n_P$  may be considered as times of full ordering in the system. Such *irregular* transients are of special interest from the point of view of nonlinear dynamics of distributed systems [1, 6, 14, 33, 34] because they may be bottlenecked by very slow, intermittent morphogenesis of spatial-temporal structures. In the next Subsections we study this collective relaxation for cases of collapse and periodic final states of TLM evolution, including an important case of lethargic transients.

### A. Collapse of RSWs

In this Subsection, we describe a typical scenario of the two-dimensional, three-level and two-channel-diffusion (2D-3L-2C) excitable system evolution leading to the full collapse of excitations (when the system reaches the single point-like spatio-temporal attractor — stable steady state with  $L_K = L_I$  for all ACs). Some typical stages of such the scenario of evolution are shown at Figure 1.

The system evolves at first stages ( $0 < n \lesssim 10^4$ ) to a labyrinthic structure which contains several RSW-nucleating domains (Figure 1,  $n = 10000$ ).

During the next stage of the evolution, the large-scale vortex is formed (Figure 1,  $n = 60000$ ). This vortex pushes the labyrinthic structure to the boundaries of the

active medium. As the result, the vortex occupies the whole active medium (Figure 1,  $n = 100000$ ).

But this giant rotating structure is unstable. Its core splits into RSWs with  $Q_T = -1$ , and at this stage the active medium is filling by such the RSWs (Figure 1,  $n = 160000$ ).

The next, relatively long stage of evolution is characterized by restless moving and strong competition of RSWs. As the matter of fact, this stage is of the "winner takes all" (WTA) type. So, at the end of this stage, the single, fully regular RSW with  $Q_T = -1$  occupies the whole active medium (Figure 1,  $n = 365000$ ).

But the core of the winner is not far from the active medium boundary. Drift of this RSW leads to collision of its core with boundary, and the core is absorbed by the latter (Figure 1,  $n = 391000$ ). The residual non-spiral autowaves gradually run out the active medium (Figure 1,  $n = 391000$ ) and excitation collapses fully at  $n = n_C = 396982$  (not shown at Figure 1).

The described scenario of excitations collapse is very typical but not the single one. Some other ways leading to collapse will be described in a forthcoming publications. Now we will consider cases of non-collapsing systems.

### B. Dynamic Stabilization, Synchronization and Coexistence of RSWs

In this Subsection, we describe a typical scenario of the 2D-3L-2C excitable system evolution leading to dynamic stabilization and coexistence of RSWs. In this case the system reaches one of its periodic spatio-temporal attractors — fully synchronized cyclic transitions ( $L_I \rightarrow L_{III} \rightarrow L_{II} \rightarrow L_I \rightarrow \dots$ ) for all ACs. Some typical stages of such the scenario of evolution are shown at Figure 2.

The system evolves at first stages ( $0 < n \lesssim 10^4$ ) to a mixed RSW-labyrinthic structure (Figure 2,  $n = 10000$ ), where nucleation of RSWs has more pronounced form, than at Figure 1 (for the same  $n = 10000$ ).

All the subsequent stages of evolution at Figure 2 differ more and more comparatively to Figure 1. Instead of formation of a giant vortex, which pushes the labyrinthic structure to the boundaries of the active medium (Figure 1), at Figure 2 we see multiple emergent RSWs ( $n = 30000$ ). These RSWs have different  $\text{sgn}(Q_T)$ , but RSWs with  $\text{sgn}(Q_T) = -1$  are prevailed here (Figure 2,  $n = 30000$ ).

As the result of such direction of evolution, at the next stage only negative-charged, well-formed RSWs with  $Q_T = -1$  occupy the whole active medium (Figure 2,  $n = 100000$ ) — instead of the single giant multi-charged vortex with  $|Q_T| > 1$  (Figure 1,  $n = 100000$ ).

Due to complex interactions between these RSWs as well as between RSWs and boundaries, the processes of revival of positive-charged RSWs take place (Figure 2,  $n = 165000$ ). In contrast to Figure 1, where strong competition between RSWs leads to WTA dynamics, compe-

tition is weakened in the case under consideration (Figure 2,  $n = 204000$ ).

Weakening of competition and low mobility of RSWs determine qualitatively another final (Figure 2,  $n = 300000$  and  $n = 987000$ ) of evolution scenario (comparatively to Figure 1). Namely, the dynamic stabilization of RSWs over the active medium is observed at  $n \approx 300000$ . Configuration of the system is precisely repeated (compare  $n = 300000$  and  $n = 987000$  at Figure 2). In other words, this is spatio-temporal limit cycle.

Fully synchronized cyclic transitions for all ACs ( $L_I \rightarrow L_{III} \rightarrow L_{II} \rightarrow L_I \rightarrow \dots$ ) for this system (Figure 2) is in contrast to collapse of excitations for the previous system (Figure 1) despite of obvious transient ordering of the spatio-temporal dynamics in the course of events for both the cases (Figure 1 and Figure 2). Note that in *both* these cases we deal with spontaneous decreasing of effective freedom degrees quantity during the evolution of the system [13].

Transient ordering of this kind, leading to collapse of excitations (Figure 1), may be interpreted as self-organized degradation (self-destruction) of the system. Final state of such the system, of course, is not self-organized, but it is still highly ordered. This kind of static order is known as freezing, because all ACs of the system occupies their ground levels (like usual particles of a physical system at absolute zero temperature).

In contrast, transient ordering, leading to dynamical stabilization of more or less complex spatio-temporal structures (Figure 2), may be interpreted as “normal” self-organization in autonomous dissipative system [13] when most degrees of freedom are “slaved” by the small rest of ones. At this self-organized state, the system reaches the end of its evolution. But the system remains unfrozen, it is still far from equilibrium and possesses obvious order in the form of spatio-temporal limit cycle (Figure 2). All parts of this system (namely RSWs) are well defined and robust (as individual dynamical spatio-temporal autostructures). At the same time, all these parts and entirely synchronized at global level (as dynamical components of the whole system demonstrating collective rhythmic motions).

Nevertheless, both these scenarios of dissipative system evolution (self-destruction at Figure 1 and normal self-organization at Figure 2) exhibit more or less fluent transition from initial spatial chaos to static (Figure 1) or dynamically stable (Figure 2) order.

A different scenario is exemplified in the next Subsection.

### C. Transient Chaos Caused by Competition of RSWs and Labyrinths

In this Subsection, we will describe another way to reach periodic state in a bounded discrete 2D-3L-2C excitable system. Formally, the final state for this new case is spatio-temporal limit cycle too (as in the case of Figure 2).

But the time of transient process is much longer here, and the longest part of the transient proceeds in the form of lethargic competition between various spatio-temporal structures.

As it was already shown in the preceding Subsection (Figure 2), spatio-temporal structures may actively interact not only between themselves, but they interact in a nontrivial manner with boundaries of the excitable medium too. The simplest case when drift of RSWs leads to collisions of their cores with boundary, and the cores are absorbed by the latter (Figure 1,  $n = 391000$ ), is not typical case for the system studied in this Subsection. On the contrary, RSWs in such the system under appropriate conditions may be “reflected” from boundaries in a strongly nonlinear manner<sup>1</sup>. Moreover, due to nonlinear processes in the surface layer, the quantity of “reflected” RSWs may exceed the quantity of primary RSWs (phenomenon of multiplication of RSWs). A detailed description of nonlinear “reflections” of RSWs and some accompanied phenomena will be published in separate paper.

Here we describe a particular, but important case of highly bottlenecked transient process, caused mainly by almost everlasting competition of RSWs and labyrinths. At Figure 3, a fragment of evolution is shown at  $n > 10^6$  for the 2D-3L-2C system, which differs from the previous system (Figure 2) only by two parameters:  $\tau_r = 44$ ;  $h = 46$  for Figure 3 (instead of  $\tau_r = 46$ ;  $h = 51$  for Figure 2). Starting pattern for the system at Figure 3 is precisely the same as at Figure 2 (so it is not shown at Figure 3).

Previous two systems (Figures 1 and 2) reach their attractors already at  $n < 5 \cdot 10^5$ . The system under consideration (Figure 3) has very long transient time, because of lethargic competition between RSWs and labyrinthic structures under conditions of regeneration (nonlinear “reflections”) and multiplication of RSWs at boundaries. There are many fine phenomena accompanied these competition. E. g., the largest RSW with  $Q_T = +1$  resided in bottom left corner of the active medium (Figure 3,  $n = 1307000$ ) moves in the *North* direction to the big labyrinth. Finally, the latter absorbs the core of the pointed RSW (Figure 3,  $n = 1322000$ ). At the same time new small RSWs are generated in the bulk active medium and regenerated (by nonlinear “reflections”) in the surface layer.

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<sup>1</sup> Reflection of RSW from boundary, as well as reflection of any other moving *dissipative* structure (autostructure, autowave) is usually considered as forbidden. This is correct, commonly speaking, only in lowest (weakly nonlinear) approximations, when the properties of surface layer are assumed as close to properties of bulk active medium. But real processes in surface layer may lead to strong nonlinear phenomena of, e. g., revival (regeneration) of RSWs or even to their duplication, triplication etc. Complex phenomena of such the kind may be considered as higher-order nonlinear interactions of RSWs with boundary, even if boundary conditions itself are “simple” (e. g. zero-flow conditions, as in the present work).

The dominant drift directions of RSWs in the system under consideration lie at *North↔South* and *East↔West* lines (this circumstance is evident when one see a movie of the system evolution). Drift of labyrinths is more complicated. In the fragment shown at Figure 3, the big labyrinth is slowly moving approximately in *North-East* direction. A new labyrinth is formed simultaneously. This new labyrinth pushes RSWs (Figure 3,  $n = 1322000$ ) etc.

The fragment of evolution shown at Figure 3 demonstrate only small part of transient phenomena observed during this system evolution. We will mention now such the typical stages of the system evolution (*not shown*<sup>2</sup> at Figure 3):

- Nonlinear “reflection” of large RSW with  $|Q_T| = 1$  from boundary accompanied by changing of  $\text{sgn}(Q_T)$  and simultaneous birth of several new small RSWs (satellites) nearby the core of large RSW ( $672000 \lesssim n \lesssim 687000$ ).

- Rotation of giant labyrinth around the domain occupied by RSWs ( $2060000 \lesssim n \lesssim 2150000$ ); erosion of this labyrinth due to RSWs activity up to almost full dominance of RSWs over active medium ( $2160000 \lesssim n \lesssim 2180000$ ); revival of labyrinth structures and resumption of competition between RSWs and labyrinths ( $2190000 \lesssim n \lesssim 2210000$ ).

- Nonlinear “reflection” of large RSW with  $|Q_T| = 1$  from boundary with saving of  $\text{sgn}(Q_T)$  ( $2309000 \lesssim n \lesssim 2324000$ , upper part of active medium). Shining examples of multiplication of RSWs at boundaries (the same interval  $2309000 \lesssim n \lesssim 2324000$ , left, right and bottom boundaries).

- Birth of self-organized pacemaker (a source of concentric autowaves) due to nonlinear processes of RSW’s core transformations in the surface layer; evolution and decay of the pacemaker ( $2965000 \lesssim n \lesssim 2980000$ , left boundary of the active medium).

- Pushing of labyrinth by a group of RSWs ( $3100000 \lesssim n \lesssim 3115000$ ) etc.

From the formal point of view, all these stages of the system evolution are only partial transient episodes on the long way to the final self-organized state. We know that such the state is a regular attractor (spatio-temporal limit cycle or at least point-like attractor) because by definition our system is autonomous, it has bounded quantity of discrete elements, each element has bounded quantity of discrete states, and time is discrete too. But we do not get concrete view of the attractor until we reach it. Shortly, “transient is nothing, attractor is all”.

From an alternative point of view, each stage of the system evolution is a day in the life of this system, and the final of evolution is of minor interest, because no new events will come to pass if an attractor is already reached. Shortly, from this point of view, “transient is

all, attractor is nothing”.

But this alternative is not the single one. Another alternative is as follows: “transient is all *that we can see*, because attractor is unachievable”.

This last alternative (for a non-CA system with continuous spectrum of states of its elementary components) was discussed in a seminal work of J. P. Crutchfield and K. Kaneko [6], which was entitled: “Are Attractors Relevant to Turbulence?”. In [28], this question was slightly reformulated in the context of our studies of CA systems: “Are Attractors Relevant to Transient Spatio-Temporal Chaos?” (turbulence in bounded, fully discrete system is no more than metaphor). The answer is “Yes” if an attractor may be reached for a reasonable time  $\tau_{\text{attr}}$  (in fully discrete and bounded system  $\tau_{\text{attr}}$  is always limited by quantity of all possible states of the system). But the answer is “No” if  $\tau_{\text{attr}}$  exceeds any possible duration of an experiment. In this case a system with transient spatio-temporal chaos cannot be distinguished from true chaotic system without additional testing.

As a matter of fact, there are some intermediate classes of phenomena “at the edge between order and chaos” which may appear in bounded discrete deterministic system with large phase space. Self-organization scenario which includes super-slow, bottlenecked, chaotic-like stages is a signature of dominance of such an intermediate class of system dynamics in numerical experiments. Really, having limited time and computer capacity, one cannot reach final self-organized state for a system with huge dimension of phase space. Such CA (or another discrete mapping with lethargic evolution) does not permit direct forecasting of the system future without direct computation. So the computed part of transient process is the single source of available information of our fully deterministic but partially determined system.

Generally, the ideas of interconnection between chaoticization, turbulence, unpredictability (at one side) and ordering, self-organization, long-time forecasting (at the opposite side) may be very fruitful at least for investigation of complex deterministic systems.

## V. CONCLUSIONS

In this work, we fulfill computer modeling of spatio-temporal dynamics in large discrete systems of three-level excitable ACs interacting by short-range 2C diffusion. Computer experiments with this 2D-3L-2C system were carried out using the cross-platform software package “Three-Level Model of Excitable System” (TLM) © 2006 S. D. Makovetskiy [31].

The most typical scenario of evolution is as follows. A robust RSW or more complicated but strictly periodic in time structure is formed being a cyclic attractor at several initial conditions — see Figure 2. This is an analog of limit cycle, i. e. regular attractor known in lumped dynamical systems. Alternatively, collapse of excitations, i. e. full freezing of the excitable system takes place —

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<sup>2</sup> Additional figures to this Preprint may be requested from the author by e-mail (see the first page of the Preprint)

see Figure 1. This is an analog of point-like attractor in lumped dynamical systems.

Long-time evolution ( $10^6 - 10^7$  iterations) of a 2D-3L-2C system with slightly changed parameters demonstrates some unusual phenomena including highly bottlenecked collective relaxation of excitations. Part of these phenomena are caused by mechanism of nonlinear regeneration of RSWs in the surface layer of an active medium. Instead of well known scenario of RSW collapse, which takes place after collision of RSW's core with absorbing boundary, we observed complicated transformations of the RSWs cores leading to nonlinear "reflections" of the RSWs from the normally absorbing boundaries or even to birth of new RSWs in the surface layer. Lethargic transient processes observed in our computer experiments are partially caused by competitions between permanently reviving RSWs and labyrinthic spatio-temporal structures. To our knowledge, phenomena of such the nonlinear "reflections" of RSWs and resulting die hard vorticity in excitable media with absorbing boundaries

were unknown earlier.

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## APPENDIX A: LIST OF ABBREVIATIONS

- 1C, 2C, ... — One-Channel, Two-Channel, ...
- 1D, 2D, ... — One-Dimensional, Two-Dimensional, ...
- 2L, 3L, ... — Two-Level, Three-Level, ...
- AC — Active Center
- CA — Cellular Automaton
- CP — Control Parameters
- GBC — Geometry and Boundary Conditions
- RSW — Rotating Spiral Wave
- TLM — Three-Level Model (of excitable system)
- ZM — Zykow-Mikhailov

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## FIGURE CAPTIONS

to the paper of S. D. Makovetskiy

“Numerical Modeling of Coexistence, Competition and Collapse  
of Rotating Spiral Waves in Three-Level Excitable Media  
with Discrete Active Centers and Absorbing Boundaries”

(figures see as separate PNG-files)

**Figure 1: Birth, evolution and collapse of RSWs with the same (by magnitude and sign) effective topological charges.** Dimensions of the active medium are  $M_X = M_Y = 300$ . Black, gray and white pixels denote ACs in excited ( $L_{\text{III}}$ ), refractory ( $L_{\text{II}}$ ), and ground ( $L_1$ ) states respectively. Starting pattern is shown at  $n = 0$ . The relaxation times of ACs are  $\tau_e = \tau_r = 50$ . The set of CPs is as follows:  $g = 1$ ;  $h = 50$ ;  $f = 3$ .

**Figure 2: Birth, evolution, dynamic stabilization and coexistence of RSWs.** Dimensions of the active medium and colors of pixels are the same as at Figure 1. Starting pattern ( $n = 0$ ) is not the same as at Figure 1, but statistical properties of both patterns are almost identical. The relaxation times of ACs and the set of control parameters are as follows:  $\tau_e = 60$ ;  $\tau_r = 46$ ;  $g = 1$ ;  $h = 51$ ;  $f = 2$ .

**Figure 3: Typical stage of transient spatio-temporal chaos (competition of RSWs and labyrinthic structures) at  $n \gg \max(\tau_e, \tau_r)$ .** Dimensions of the active medium and colors of pixels are the same as at Figures 1 and 2. Starting pattern is precisely the same as at Figure 2 (and not shown here). The relaxation times of ACs and the set of CPs are as follows:  $\tau_e = 60$ ;  $\tau_r = 44$ ;  $g = 1$ ;  $h = 46$ ;  $f = 2$ .

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